Technical Comment

Potential Flow around a Rotating Blade

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THIS Comment discusses W. R. Sears' article "Potential Flow Around a Rotating Cylindrical Blade."[1] First, the problem as formulated by Sears does not have a unique solution. However, the potential function he obtained for a circular cylinder is identical to the potential flow about a rotating ellipsoid in the limiting case of infinite length. Second, a blade of finite length has an additional component of spanwise flow. For example, in the case of the circular cylinder the spanwise velocity is zero on the surface of the cylinder whereas the rotating ellipsoid has a nonzero spanwise velocity as indicated in Fig. 2. This component may be important when the potential flow is used to calculate laminar boundary layer flow on rotating blades, since, in the case of the rotating ellipsoid, it is of the same order of magnitude as the crossflow velocities encountered in the flat plate case.

Let us examine the first comment. Sears considers the problem of an infinitely long cylinder rotating steadily about an axis normal to it in a perfect incompressible fluid otherwise at rest. Let ω denote the angular velocity and let x,y,z denote Cartesian coordinates rotating with the blade, where y is measured outward along the blade from the center of rotation, z is measured along the axis of rotation, and x forms the third axis of a right-hand system. He reduces the problem to the following: determine the function ϕ which satisfies the equation

$$\Delta\phi = 0$$

and which satisfies the boundary conditions

$$l(\phi_x + \omega_y) + n\phi_z = 0 \text{ on } C \tag{1}$$

where C is the contour of the cylinder cross section and (l,m,n) is the normal to C, and

$$\phi_x,\phi_y,\phi_z \rightarrow 0 \text{ as } x \rightarrow \infty \text{ and/or } z \rightarrow \infty \tag{2}$$

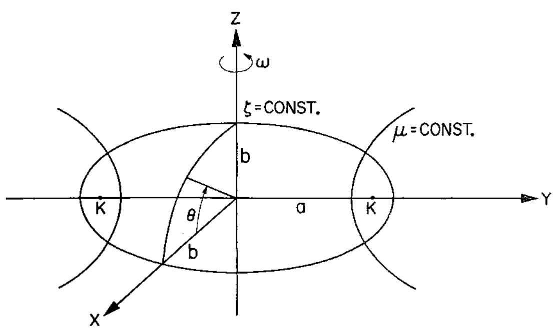

Fig. 1 Coordinate system.

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Next Sears states that these are exactly the boundary conditions for the plane flow produced by motion of the infinite cylinder in the $-x$ direction with speed ωy, and the differential equation for the potential of that flow is the same as for ϕ if $\phi_{yy} = 0$. He proceeds to solve this two-dimensional problem and states that it is the solution to the three-dimensional one.

There are two objections to that procedure. First the condition $\phi_y \rightarrow 0$ as $x \rightarrow \infty$ and/or $z \rightarrow \infty$ is a boundary condition in the third dimension. More importantly, to determine the potential ϕ one is not allowed to set $\phi_{yy} = 0$ but must also allow those solutions for which $\phi_{yy} \neq 0$. The unique solution is then determined by imposing proper boundary conditions. However, the problem under consideration is not properly posed because there is no boundary condition on ϕ_y as $y \rightarrow \infty$.

In the case of a circular cylinder of radius one, Sears has found the solution

$$\phi = \omega y s^{-1} \cos\theta$$

where $s = (x^2 + z^2)^{1/2}$ and $\theta = \tan^{-1}(z/x)$. However, there are additional solutions for which $\phi_{yy} \neq 0$ and which satisfy the boundary conditions (1) and (2) as well as the requirements that $\nabla\phi$ be single-valued and posses certain symmetries. All these solutions are possible motions of the fluid as the problem has been posed.

Let us resolve this problem of which solution to pick by examining the case of an ovary (i.e., prolate) ellipsoid rotating about an equatorial axis. This problem is well-posed and the solution is found in Ref. 2. We change his notation slightly to conform with Sears' choice of Cartesian axes (see Fig. 1).

$$y = k\mu\zeta,\ x = \sigma\cos\theta, z = \sigma\sin\theta,$$

$$\sigma = k(1 - \mu^2)^{1/2}(\zeta^2 - 1)^{1/2}$$

The surfaces ζ = constant, μ = constant are confocal ellipsoids and hyperboloids of two sheets, respectively, the common foci being the points $(\pm k,0,0)$. The value of ζ may range from 1 to ∞, while μ lies between ± 1. The coordinates μ,ζ,θ form an orthogonal system. The surface of the rotating ellipsoid is $\zeta = \zeta_0$ and a and b are the polar and

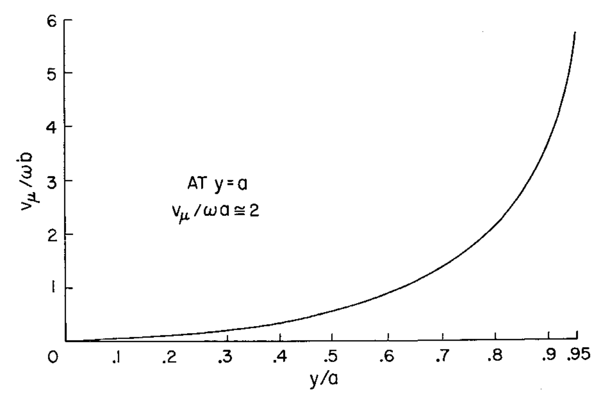

Fig. 2 Spanwise flow.

equatorial radii, respectively. Hence, $a = k\zeta_0$ and $b = k(\zeta_0^2 - 1)^{1/2}$.

The potential ϕ is given by

$$\phi = A\mu(1 - \mu^2)^{1/2}(\zeta^2 - 1)^{1/2} \times \{\tfrac{3}{2}\zeta \log[(\zeta + 1)/(\zeta - 1)] - 3 - [1/(\zeta^2 - 1)]\} \cos\theta \quad (3)$$

A is determined by the surface boundary condition

$$\partial\phi/\partial\zeta = -k^2\omega[1/(\zeta^2 - 1)^{1/2}]\mu(1 - \mu^2)^{1/2} \cos\theta \text{ at } \zeta = \zeta_0$$

Now we fix $b = 1$ and take the limit as $k \to \infty$ to find

$$\phi \to \omega y s^{-1} \cos\theta$$

which is the Sears' solution.

Next we examine the component of spanwise flow that a blade of finite length has, but which is neglected by assuming infinite length. In the case of a circular cylinder, the spanwise flow on the surface is zero, as was proven by Sears.[1] In Fig. 2, we plot the μ velocity component along the leading edge of the rotating ellipsoid for $b/a = 1/100$. Here

$$v_\mu = v_{\mu p} + v_{\mu r} \quad (4)$$

with

$$v_{\mu p} = -c\omega b[(1 - 2\mu^2)/(\zeta_0^2 - \mu^2)^{1/2}],$$
$$v_{\mu r} = \omega b\zeta_0/(\zeta_0^2 - \mu^2)^{1/2}$$

$v_{\mu p}$ is the μ component of the gradient of the potential ϕ in Eq. (3) and $c \simeq 1$ for $b/a = 1/100$. $v_{\mu r}$ is the μ velocity component because of (ζ,μ,θ) being a rotating coordinate system. v_μ is a good approximation of the y velocity component for $y \leq 0.95a$ and $c \simeq 1$.

For an ellipsoid translating at velocity V, v_μ is a function similar to (4), the difference being that the numerator has the form $d(V/a)b\mu$, with $d \cong 2$ for $b/a = 1/100$. Hence, with $v_\mu/[(V/a)b]$ as the ordinate, a curve similar to Fig. 2 holds for a translating ellipsoid.

Finally note that if an infinitely long cylinder is used, there is no way to equate coordinates, and hence velocities, in the y direction to actual spanwise distances on a finite blade, whereas a model of finite length easily relates the two.

References

[1] Sears, W. R., "Potential Flow Around a Rotating Cylindrical Blade," *Journal of Aeronautical Sciences*, Vol. 17, No. 3, March 1950, p. 183.

[2] Lamb, H., *Hydrodynamics*, Cambridge University Press, New York, 1932, p. 142.

Announcement: 1970 Author and Subject Indexes

The indexes of the four AIAA archive journals (*AIAA Journal*, *Journal of Spacecraft and Rockets*, *Journal of Aircraft*, and *Journal of Hydronautics*) will be combined and mailed separately early in 1971. Subscribers are entitled to one copy of the index for each subscription which they had in 1970. Extra copies of the index may be obtained at $5 per copy. Please address your request for extra copies to the Circulation Department, AIAA, Room 280, 1290 Avenue of the Americas, New York, New York 10019.

The 1968 combined index (including the indexes of the 1967 issues of the *Journal of Hydronautics*) was reprinted, and a few copies are still available, at $5 per copy, on a first-come, first-served basis. Orders should be sent to the Circulation Department at the above address and must be received by February 15, 1971. A very few copies of the 1969 index are available on the same basis as the 1968 index.

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